


1993

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Giles, Timothy L., "Life Insurance Applications of Recursive Formulas" (1993). *Journal of Actuarial Practice 1993-2006*. 167.
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Life Insurance Applications of Recursive Formulas

L. Timothy Giles*

Abstract¹

This paper discusses several practical applications of recursive formulas:

- a) Traditional whole life—As an introduction, the well-known relationship between successive terminal reserves is reviewed. Recursive formulas are developed to calculate the reserves and the premiums;
- b) Universal life—Recursive formulas are used both for the calculation of target premiums and reserves. Consideration is given to the TEFRA corridor;
- c) Paid-up rider—A participating single premium rider that provides a level death benefit can be devised using an inherent one year term benefit. Recursive functions are used to determine the premium that precisely matures the rider.

Because the APL programming language is particularly amenable to recursive formulas, a few sample APL programs are provided.

Key words: *APL, TEFRA corridor, universal life, paid-up rider*

1 Introduction

A recursive formula is one where the current result is generated from previous results once the starting values are given. Essentially, a recursive formula is a difference equation with known starting values. Some formulas are special linear difference equations that, when added, condense to the first and last values of the recursion. Thus, if the starting value is known (usually zero), the ending value (usually the maturity amount) and all the intermediate values can be derived easily.

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¹ This paper is an extension of the author's paper entitled "The Practical Use of Recursive Functions" that appeared in ARCH, 1993.3.

Actuaries are familiar with such formulas, especially in the area of reserve and asset share calculations; see, for example, Jordan (1991) and Bowers *et al.* (1986). Shiu (1987) and Seah and Shiu (1987) elegantly present recursive formulas in their discussions of papers by Berin and Lofgren (1987) and Eckley (1987), respectively.

My interest in recursive formulas began in 1985 with the problem of finding the target premium for universal life policies. What level deposit (along with credited interest and mortality and expense charges) would mature the policy exactly? At that time I used trial and error to obtain the target premiums. A much better method, however, later was published by Eckley (1987).² What these pioneers discovered is that by defining a transformed mortality rate $Q' = Q / (Q + 1)$, the traditional commutation functions could be used (with Q' replacing Q) to calculate premiums and account values directly. These modified commutation functions are called *transformation functions*.

When the accumulation formula changes due to the TEFRA³ corridor (if the cash value becomes too high in relation to the death benefit, a higher minimum death benefit is invoked), however, a different formula for Q' must be used. One, theoretically, could switch functions at the duration of change. I believe, however, that recursive formulas offer a better solution. Recursive formulas also can be used to perform the intricate calculation of paid-up riders that fund a benefit with a combination of one year term and paid-up insurance.

2 Introduction to the Method

Following Shiu (1987), define a first order linear recursive (difference) equation as

$$x_{k+1} = a_k x_k + b_k \quad k = 0, 1, 2, \dots \quad (1)$$

where x_0 , $\{a_k\}$, and $\{b_k\}$ are known. Let

$$s_{k+1} = a_0 a_1 \dots a_k = \prod_{r=0}^k a_r \quad (2)$$

with $s_0 = 1$.

Dividing both sides of equation (1) by s_{k+1} yields

² I am also familiar with a similar (though unpublished) work by Wesley C. Green. Mr. Green currently is director of Actuarial Systems and MIS at Phoenix Home Life Mutual Insurance Company in Hartford, CT.

³ Tax Equity and Fiscal Responsibility Act (1982).

$$\frac{x_{k+1}}{s_{k+1}} = \frac{x_k}{s_k} + \frac{b_k}{s_{k+1}}$$

i.e.,

$$\Delta \left(\frac{x_k}{s_k} \right) = \frac{b_k}{s_{k+1}}. \quad (3)$$

Summing equation (3) from $k=0$ to $k=n-1$ yields

$$x_n = \left(x_0 + \sum_{k=0}^{n-1} \frac{b_k}{s_{k+1}} \right) \times s_n. \quad (4)$$

Thus, once the starting value x_0 is determined, the n^{th} term can be obtained directly without explicitly computing the intermediate terms.

3 Traditional Whole Life

Here is the procedure for deriving a recursive formula for a traditional whole life reserve with a death benefit of 1:

a) **Establish the succession rule.**

You must know precisely the mathematical relationship between the reserve at t and that at $t+1$. For example, for traditional whole life with face value t , it is well known (see, for example, Bowers *et al.*, Chapter 7, Section 8) that the succession rule for the reserves is:

$$({}_tV + P) \times (1+i) = {}_{t+1}V \times (1-Q_t) + Q_t \quad (5)$$

where P is the net level premium, ${}_tV$ is the net premium terminal reserve at time t , Q_t is the valuation mortality rate at age $x+t$ (x is the issue age), and i is the valuation interest rate.

b) **Cast the succession rule into linear form.**

Put equation (5) in linear form (as in equation (1)),

$${}_{t+1}V = {}_tV \times \frac{(1+i)}{(1-Q_t)} + P \times \frac{(1+i)}{(1-Q_t)} - \frac{Q_t}{(1-Q_t)}. \quad (6)$$

c) **Compute the compounding element from issue to maturity.**

The compounding element F_k , as in equation (2), gives

$$F_{k+1} = \prod_{j=0}^k \left[\frac{(1+i)}{(1-Q_j)} \right].$$

Notice that $1/F_k$ is a discount factor, i.e.,

$$\frac{1}{F_k} = (1+i)^{-k} {}_kP_x = \frac{D_{x+k}}{D_x}.$$

- d) **Divide both sides of the succession rule by F_{t+1} and place the result in finite difference form as in equation (3).**

$$\begin{aligned} \Delta \left(\frac{{}_tV}{F_t} \right) &= \left[\frac{P \times ((1+i)/(1-Q_t))}{F_{t+1}} \right] - \left[\frac{Q_t/(1-Q_t)}{F_{t+1}} \right] \\ &= \frac{P}{F_t} - \frac{Q_t/(1-Q_t)}{F_{t+1}}. \end{aligned}$$

- e) **Sum both sides from issue to maturity.**
Let m be the number of years from issue to maturity. Note that ${}_mV = 1$ and ${}_0V = 0$.

$$\frac{{}_mV}{F_m} = \sum_{k=0}^{m-1} \left[\frac{P}{F_k} - \frac{Q_k/(1-Q_k)}{F_{k+1}} \right]. \quad (7)$$

- f) **Solve for P .**
From equation (7),

$$P \times \sum_{t=0}^{m-1} \frac{1}{F_t} = \frac{1}{F_m} + \sum_{t=0}^{m-1} \frac{{}_t v Q_t}{F_t} \quad (8)$$

where $v = (1+i)^{-1}$. Note that equation (8) can be rearranged as follows

$$\frac{{}_mV}{F_m} - \frac{{}_0V}{F_0} = P \times \sum_{t=0}^{m-1} \frac{1}{F_t} - \sum_{t=0}^{m-1} \frac{{}_t v Q_t}{F_t}$$

which essentially states that

Maturity value – Issue value = Premiums – Claims,

with all terms discounted to issue.

- g) **Generate the intermediate values from the succession rule and the premium.**

From equation (4), it follows that

$${}_tV = F_t \times \sum_{k=0}^{t-1} \frac{P - vQ_k}{F_k}.$$

Notice that we have discounted all terms to issue, then accumulated.

4 Universal Life

This demonstration assumes a level death benefit of 1. The actuarial starting point for universal life mathematics is equation (5), the formula connecting successive terminal reserves for conventional whole life insurance:

$$({}_tV + P)(1+i) = {}_{t+1}V + Q_t \times (1 - {}_{t+1}V).$$

Here $1 - {}_{t+1}V$ is called the *net amount at risk*, and $Q_t \times (1 - {}_{t+1}V)$ is called the *mortality charge*.

Three sources of difficulties have to be overcome:

- The mortality charge, being an expense, is payable at the beginning of the period, whereas the whole life reserve formula assumes deaths occur at the end of the year.
- The net amount at risk is defined contractually. The traditional definition cannot be used because the end of the year reserve (or cash value) is not known at the time the mortality charge is due, which is at the beginning of the time period. A common approximation⁴ is to use

$${}_{t+1}V \doteq ({}_tV + P)(1+i). \quad (9)$$

⁴ An alternative approximation is to include the mortality charge in the approximation, i.e.,

$${}_{t+1}V \doteq ({}_tV + P)(1+i) - Q_t \times (v - ({}_tV + P)).$$

Discounting the approximate mortality charge at interest results in $Q_t \times (v - ({}_tV + P))$ being the current mortality charge. Note that i is the guaranteed rate because the current rate is unstable.

- c) The crediting and charging is done monthly, not annually as in the whole life reserve.

Monthly interest and mortality charges can be used directly or an algorithmic adjustment to annual processing can shorten the computation time slightly. Eckley explains this, noting that the mortality charge and the interest credit are constant for all 12 months of a policy year, at least in sales illustrations. The interested reader is referred to Eckley's paper for the formulas. The recursive formulas will work if monthly mortality is used, although the vector will be 12 times as large. A monthly interest rate is installed easily.

An accurate target premium also must include expense charges. The succession rule must be based upon the actual administrative processing (typically done by a mainframe computer). If that processing is in place, the actuary has to try to mimic the routines used on the mainframe instead of establishing theory.⁵

There is, of course, a different succession rule when the TEFRA corridor is in effect. To qualify as a life insurance product in the United States and receive the attendant tax benefits, there must be a minimum relationship between the death benefit and the cash surrender value. This is called the cash value (or TEFRA) corridor and is 250 percent for attained ages 40 and under and gradually reduces to 100 percent at age 95. There is no sound actuarial basis for these ratios; they are simply products of the U.S. Congress.

The most common industry response to this requirement is to include a contractual benefit setting the death benefit equal to the minimum of the face amount or the TEFRA corridor multiplied by the cash surrender value. This response in turn generates a complex NAIC⁶ reserve mandate that in effect requires the immediate funding of any projected triggering of the corridor. If the cash surrender value at the valuation date is such that its accumulation along with future guaranteed maturity premiums will cause the death benefit to exceed

⁵ An actuary, at an earlier time, may have established the processing rule. There are details (particularly when the minimum death benefit is in effect) such as the definition of the net amount of risk that are subject to choice. The practicing actuary then has to mimic the succession rule that is in place.

⁶ The National Association of Insurance Commissioners (NAIC) is an association consisting of state insurance commissioners. The NAIC drafts model laws and recommends their adoption by state legislatures. The NAIC has no legal authority to force states to enact its recommendations.

the face amount, the present value of the excess is added to the reserve. In some cases the calculation has to jump between the two rules, but usually not more than once.

The succession rule for universal life (using the approximation to ${}_{t+1}V$ given in equation (9)) is

$${}_{t+1}V = ({}_tV + P) \times (1 + i) - Q_t \times (v - {}_tV - P) \times (1 + i).$$

For the TEFRA corridor, the death benefit is the cash value times $(1 + \Theta_t)$, where $\Theta_t \geq 0$. The cash value at the end of period $t+1$, where death is assumed to occur, is ${}_{t+1}CV$, where ${}_{t+1}CV = ({}_tV + P) \times (1 + i)$. Thus, the net amount at risk at that time is

$$(1 + \Theta_t)({}_tV + P)(1 + i) - ({}_tV + P)(1 + i) = \Theta_t \times ({}_tV + P)(1 + i).$$

The annual mortality charge is paid at the beginning of the year. Hence:

$${}_{t+1}V = ({}_tV + P) \times (1 + i) - Q_t \times ({}_tV + P) \times \Theta_t.$$

5 Paid-Up Insurance Rider

The conventional paid-up addition is a single premium participating product. Each dividend then buys another portion of paid-up insurance, which is itself participating. The effect is an increasing amount of paid-up insurance.

A variation of this effect is a level death benefit participating single premium whole life policy. The facilitating device is to divide the death benefit into a one year term portion and a paid-up portion. The dividend first pays for the next year's term insurance with the same death benefit; any remaining dividend buys a paid-up participating whole life benefit. If the dividend is not sufficient to pay for the term portion, some of the paid-up could be surrendered and the face value of the term increased. If reduced dividends continued, there will be a time when the death benefit would have to be decreased. The initial premium rate is designed to pay for both the term portion and the paid-up portion, thus avoiding paying for the term in arrears.

Let P = the single premium. It buys a combination of paid-up, PU , and a one year term insurance determined from:

$$PU \times A_x + (1 - PU) \times c_x = P$$

i.e.,

$$P = (A_x - c_x) \left[\frac{1}{F_m} - \sum_{t=0}^{m-1} \frac{(DIV_{t+1} - c_{x+t+1})}{(A_{x+t+1} - c_{x+t+1})F_{t+1}} \right].$$

The result is the exact premium that will mature the policy if dividends are paid as projected; a daunting task by trial and error.

There are constraints that should be noted. If the paid-up amount were to exceed 1, future premiums would not be accepted. Also, negative paid-up amounts are not admissible.

6 Recursion Formulas in APL

APL programming language is well-suited to recursive formulas, as partial products are generated easily. A good APL technique for accumulating nonlevel payments is to discount all of the payments to the present, then accumulate this lump sum. The same process is used with recursive formulas.

Refer to the first practical example of traditional whole life,

$$F_t = \times \backslash (1 + i) \div (1 - Q_t) \text{ and } F_m = 1 \uparrow F_t.$$

In APL,

$$P = \{((1 \div F_m) + \div (Q_t \div (1 - Q_t)) \div F_t)\} \div (-1 \downarrow 1, V1_t)$$

The nonlevel payment accumulation technique described above can be used for intermediate values of the traditional whole life reserve:

$${}_tV = F_t \times (+ \backslash P \div -1 \downarrow 1, F_t) - + \backslash (Q_t \div (1 - Q_t)) \div F_t$$

7 Conclusion

For the past several decades, commutation functions have served actuaries well. The more complex products of today, however, call for new techniques. Recursive formulas may be the answer. They serve best where the calculation involves a trajectory to a target. Even whole life can be viewed as finding a premium to mature the policy. The intermediate values, the reserves, follow easily. The tradition

of valuations made with a table of stored reserve factors can be improved. The reserves can be calculated on an as needed basis (which is the way universal life reserves must be calculated). Commutation functions also could be used to calculate reserves as needed. Recursive formulas provide a second way. I hope future actuaries are as comfortable with recursive formulas as present actuaries are with commutation functions.

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